Interpretation of the topological terms in gauge system

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Abstract

We provide an alternative interpretation for the topological terms in physics by investigating the low-energy gauge interacting system. The asymptotic behavior of the gauge field at infinity indicates that it traces out a closed loop in the infinite time interval: $(-\infty,\ +\infty)$. Adopting Berry's argument of geometric phase, we show that the adiabatic evolution of the gauge system around the loop results in an additional term to the effective action: the Chern-Simons term for three-dimensional spacetime, and the Pontrjagin term for the four-dimensional spacetime.

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As the property uniquely appearing in quantum theory, topology has been acquiring much attention in both condensed matter and particle physics. The topological term for example the Pontrjagin term was first introduced in gauge field theory to resolve the anomaly in triangle graphs that breaks down the usual Ward-Takahashi identity of the chiral current[1]. It was found later that the topological term yields the *instanton* solution[2], which gives the local minima of the Yang-Mills gauge field action and brings in an interesting connection between particle physics and spacetime topology. Recently, Witten[3] introduced topological quantum field theory, which emerges as possible realization of general coordinate invariant symmetries.

One notices that in the above quantum theories, the topological term is always put in the action *by hand*, not based on a dynamical consideration. In this Letter, we attempt to give an interpretation of the topological term alternatively by exploring the implication of geometric phase in the path integral formalism of low-energy gauge interaction.

As a powerful method of quantizing quantum theory, the path integral method was developed by Feynman[4], based on Dirac's intuition[5] that the transition amplitude of a quantum system between two states $|\alpha, t\rangle$ and $|\alpha', t'\rangle$ is proportional to the phase factor in terms of the classical action of the same system ($\hbar = c = 1$):

$$\langle \alpha', t' | \alpha, t\rangle \propto e^{iS(t',t)}. \tag{1}$$

The path integral has developed into a functional integral approach to quantum field theory, which not only yields a simple, covariant quantization of complicated systems with constraints, such as gauge theory, but also leads to a deep understanding to some basic assumptions of quantum theory.

However, the ordinary path integral method needs improving in dealing with some quantum mechanical systems, for example, the adiabatically evolving systems containing geometric phases. To see this clearly, we recapitulate the basic idea of the geometric phase proposed by Berry[6]. For a system in which the Hamiltonian H evolves adiabatically with parameters $\mathbf{R} \equiv \mathbf{R}(t)$, and has a *discrete* spectrum: $H(\mathbf{R})|n, \mathbf{R}\rangle = E_n(\mathbf{R})|n, \mathbf{R}\rangle$. The state

of the system, determined by the Schrödinger equation:

$$H(\mathbf{R})|\Psi_n(t)\rangle = i\frac{\partial}{\partial t}|\Psi_n(t)\rangle, \tag{2}$$

is solved as: $|\Psi_n(t)\rangle = e^{i[\gamma_n(t)+\gamma'_n(t)]}|n, \mathbf{R}(t)\rangle$, where $\gamma_n(t) = i\int_0^t \langle n, \mathbf{R}|\partial_{t'}|n, \mathbf{R}\rangle dt'$, and $\gamma'_n(t) = -\int_0^t E_n(\mathbf{R}(t'))dt'$. If $\mathbf{R}$ executes a closed loop: $\mathbf{R}(T) = \mathbf{R}(0)$, $\gamma_n(T)$ is expressed alternatively as:

$$\gamma_n(T) = \oint \mathbf{A}(\mathbf{R}) \cdot d\mathbf{R}, \tag{3}$$

where $\mathbf{A}(\mathbf{R}) \equiv i\langle n, \mathbf{R}|\nabla_{\mathbf{R}}|n, \mathbf{R}\rangle$ is called Berry's potential. It was shown by Simon[7] that the above $\gamma_n(T)$ is attributed to the holonomy in the parameter space, thereby called *geometric phase*.

For the above cyclic evolutionary system, the transition amplitude of the states after the parameter $\mathbf{R}$ traces out a closed loop is obtained to be:

$$\langle\Psi_n(T)|\Psi_n(0)\rangle = e^{i[\gamma_n(T)+\gamma'_n(T)]}. \tag{4}$$

Comparing this result with Eq. (1), we see immediately that the classical action includes the dynamic phase only. This can be understood from two aspects: (i) The topological structure of quantum theory does not show up in the classical dynamics; (ii) Geometric phase as the quantity of one-order time derivative does not contribute to the usual Lagrangian equation of motion and the classical action either.

Let us further look at the gauge system with infinite number of degrees of freedom, which is usually treated by the perturbation theory in the interaction picture based on the adiabatic approximation. To be consistent with the boundary requirement of spacetime topology, the gauge field $\mathbf{a}(\mathbf{x}, t)$ (the temporal gauge is chosen: $a_0(\mathbf{x}, t) = 0$) generally has the following asymptotic behavior at infinity[8]:

$$\begin{cases} \mathbf{a}(\mathbf{x}, t) \to 0, \quad \text{for} \quad t \to \pm\infty; \\ \mathbf{a}(\mathbf{x}, t) \to \nabla g(x), \quad \text{for} \quad |\mathbf{x}| \to \infty. \end{cases} \tag{5}$$

These conditions further suggest that for sufficiently low-energy gauge interaction such that the creation and annihilation of particles are negligible, the gauge field $\mathbf{a}(\mathbf{x}, t)$ can be taken as the parameter space. Then the gauge system undergoes a *cyclic evolution* from $t \to -\infty$ to $t \to +\infty$. A question rises immediately:

How to take into account the effect of cyclic evolution in the above low-energy gauge interacting system?

Before a further discussion to the above question, we would like to make a digression to mention a recent discovery by Newton[9]: For a quantum mechanical system with continuous spectra for instance the scattering case, Newton introduced a so-called noninteraction picture to show that the system presents a cyclic change with time t from $-\infty$ to $+\infty$. The geometric phase factor is then proved to be nothing but the S matrix. We know that the S matrix in the scattering theory is easily formulated in terms of the path integral. This implies that the geometric phase factor can be described by the path integral. We infer further that for the above gauge system presenting a cyclic evolution, Berry's argument on geometric phase probably shows its own $effect$ in the path integral formalism of the system.

The purpose of this Letter is to approach the above $effect$. We first explain why we prefer the gauge system: (i) The importance of gauge theory in the description of the elementary particles and forces; (ii) It has been shown that gauge structure appears in the geometric phase[10]. Therefore, it would be interesting to investigate the role this gauge structure plays in the gauge theory. Without loss of generality, we consider the Abelian gauge interacting system with the action:

$$S = \int d^4x \left[i\bar{\Psi}(\gamma^\mu D_\mu + im)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right], \tag{6}$$

where $\Psi(x)$ is the fermion field with mass m, $D_\mu = \partial_\mu + iea_\mu$ and $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. We can not directly follow either Berry's calculation or Newton's noninteraction picture, because: (i) the above gauge system is unbounded and has continuous spectrum, (ii) quantum fields are described by infinite dimensional Hilbert space. In the present Letter, we are not going to get involved in the complication of the path integral formalism of the adiabatically evolving

system. Instead, we provide a simple way outlined as follows.

As we know that the gauge field $a_\mu(x)$ in above action is in fact a quantum object. However, if we take into account the geometric phase of the gauge system, $a_\mu(x)$ should be taken as the classical parameter (extension of geometric phase from classical to quantum will be discussed later). For this purpose, we consider only the case of sufficiently low-energy interaction, where the creation and annihilation of particles are negligible, and the gauge field can be treated as the parameters for the first order approximation. Under the adiabatic approximation, the equation of motion of fermion field is in nearly free-form, which is conveniently handled in momentum space. Moreover, since the gauge field is taken as the parameter, each mode of the fermion field in momentum space is independent of others, thus can be effectively treated by quantum mechanics. For simplicity in description, we neglect the mass term of the fermion field. Then the Hamiltonian of each mode in momentum space is written as:

$$h(\mathbf{a}) = -\alpha_i(p_i + ea_i), \tag{7}$$

where $\alpha_i = \sigma_i \bigotimes \sigma_1, i = 1, 2, 3$, σ_i are the Pauli matrices, and the gauge condition $a_0 = 0$ is chosen. With these preparations, we now compute the total effect of the geometric phase in the above gauge system through two steps:

First, since the gauge field in Eq. (7) satisfies the asymptotic condition: $\mathbf{a}(t \to -\infty) = \mathbf{a}(t \to +\infty) = 0$, namely, the gauge field executes a closed loop with respect to the infinite time interval: $t \in (-\infty, \ +\infty)$, each mode of the fermion field in momentum space will acquire a geometric phase after the gauge field traces out the loop. Fortunately, this geometric phase can be directly obtained by applying Berry's method to $h(\mathbf{a})$. For the eigenvalue equation in parameter space: $h(\mathbf{a})|\psi(\mathbf{a})\rangle = E(\mathbf{a})|\psi(\mathbf{a})\rangle$, we choose only the positive energy solution: $E_+(\mathbf{a}) = \sqrt{(\mathbf{p} + e\mathbf{a})^2}$, since the states are unbounded, where eigenkets are doubly-degenerate:

$$|\psi_+(\mathbf{a})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin\theta e^{-i\phi} \\ -\cos\theta \\ 0 \\ -1 \end{pmatrix}, \quad |\psi_+(\mathbf{a})\rangle' = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos\theta \\ -\sin\theta e^{i\phi} \\ 1 \\ 0 \end{pmatrix}, \tag{8}$$

where θ and ϕ are polar and azimuthal angles of the vector $\mathbf{q} \equiv \mathbf{p} + e\mathbf{a}$, respectively. Then for each mode in momentum space, the geometric phase of the ket $|\psi_+(\mathbf{a})\rangle$ is[11]:

$$\gamma_+(\mathbf{p}) = e \int_{-\infty}^{+\infty} dt \;\; A_i(\mathbf{p} + e\mathbf{a}) \;\; \dot{a}_i, \tag{9}$$

where $A_i(\mathbf{p} + e\mathbf{a}) = i\langle\psi_+(\mathbf{a})|\frac{\partial}{\partial q_i}|\psi_+(\mathbf{a})\rangle \equiv A_i$ are given by:

$$A_1 = -\frac{q_2}{2\mathbf{q}^2}, \quad A_2 = \frac{q_1}{2\mathbf{q}^2}, \quad A_3 = 0, \tag{10}$$

and satisfy the relation

$$\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} = \frac{q_3}{\mathbf{q}^4} q_k \epsilon_{ijk} + \pi\delta(q_1, q_2)\Delta(q_3)\epsilon_{ij3}, \tag{11}$$

where the function $\Delta(q_3) = 1$ for $q_3 = 0$; otherwise, $\Delta(q_3) = 0$. Under the low-energy approximation, we expand $A_i(\mathbf{p} + e\mathbf{a})$ with respect to the gauge field up to the first order: $A_i(\mathbf{p}+e\mathbf{a}) \approx A_i(\mathbf{p})+ea_j\partial A_i(\mathbf{p})/\partial p_j$. Taking this result into Eq. (9), then using the relation (11) and integrating in parts, we rearrange $\gamma_+(\mathbf{p})$ into:

$$\gamma_+(\mathbf{p}) = -\frac{e^2}{2} \int_{-\infty}^{+\infty} dt \;\; \left[\frac{p_3}{\mathbf{p}^4} p_k \epsilon_{ijk} + \pi\delta(p_1, p_2)\Delta(p_3)\epsilon_{ij3}\right] a_j \dot{a}_i. \tag{12}$$

Second step, the geometric phase in the configuration space is obtained through a Fourier transformation to $\gamma_+(\mathbf{p})$:

$$\gamma_+(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \;\; \gamma_+(\mathbf{p}) \;\; e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{13}$$

Since there are infinite number of degrees of freedom in the system, we should integrate over the whole space to get to the total contribution of the geometric phases to the gauge system:

$$\Gamma_+ = \int d^3\mathbf{x} \;\; \gamma_+(\mathbf{x}). \tag{14}$$

Γ_+ is called the *geometric term*.

There are two remarks addressed in order:

(i) The above definition of Γ_+ is different from the definition of Γ_- in Ref. [12], where the electrons are restricted in two-dimensional Dirac sea, thus have only the bounded states. However, the gauge system we consider is unbounded, which can be realized by the low-energy scattering experiment of electrons and photons. We notice that the Fourier transformation was overlooked in Ref. [12], even though it does not yield any difference between the result of Ref. [12] and that of ours as shown below.

(ii) The second term in the expression of $\gamma_+(\mathbf{p})$ is in fact a projection from three-dimensional space to two dimensions. It should be treated independently in Γ_+ through the two-dimensional Fourier transformation:

$$\Gamma_+^2 = \frac{1}{(2\pi)^2}\int dt\ d^2\mathbf{x}\int d^2\mathbf{p}\ \left[\frac{-e^2}{2}\pi\delta(p_1,p_2)\epsilon_{ij}a_j\dot{a}_i\right]\ e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{15}$$

where $i,j=1,2$. Γ_+^2 is reduced to be:

$$\Gamma_+^2 = -\frac{e^2}{8\pi}\int dt\ d^2\mathbf{x}\ \epsilon_{ij}\ a_j\dot{a}_i, \tag{16}$$

which is exactly the usual Chern-Simons action under the *temporal* gauge condition: $a_0=0$.

We now deal with the remaining term in Γ_+, which is expressed as

$$\Gamma_+^3 = \frac{ie^2}{16\pi^3}\int dt\ d^3\mathbf{x}\ \epsilon_{ijk}a_j\dot{a}_i\ \frac{\partial}{\partial x_k}f(\mathbf{x}), \tag{17}$$

where

$$f(\mathbf{x}) = \int d^3\mathbf{p}\ \frac{p_3 e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^4}. \tag{18}$$

It is evident that the above integral (18) is infrared divergent. Fortunately, this divergence can be resolved by restoring the mass m to the fermions, which leads to the replacement: $\mathbf{p}^4 \longmapsto\ (\mathbf{p}^2+m^2)^2$ in the relevant term of Eq. (18). Then $f(\mathbf{x})\longmapsto f(\mathbf{x},m)$, and Γ_+^3 is re-defined as: $\Gamma_+^3=\lim_{m\to 0}\Gamma_+^3(m)$. A careful calculation gives neatly: $f(\mathbf{x},m)=i\pi^2\ e^{-m|\mathbf{x}|}$. Therefore,

$$\Gamma^3_+(m) = \frac{-e^2}{16\pi} \int dt\ d^3\mathbf{x}\ \epsilon_{ijk} a_j \dot{a}_i\ \frac{\partial}{\partial x_k} e^{-m|\mathbf{x}|}. \tag{19}$$

After integrating by parts Eq. (19), we neglect the surface term since: $e^{-m|\mathbf{x}|} \to 0$, for $x_k \to \pm\infty$ (m is finite at this moment). Then choosing the limit $m \to 0$, we get to:

$$\Gamma^3_+ = \frac{e^2}{16\pi} \int dt\ d^3\mathbf{x}\ \epsilon_{ijk}\ \left[\partial_k a_j\ \dot{a}_i + a_j\ \partial_k \dot{a}_i\right], \tag{20}$$

which is further arranged into the following form after integrating by parts the time derivative of the second term in Γ^3_+:

$$\Gamma^3_+ = -\frac{e^2}{8\pi} \int dt\ d^3\mathbf{x}\ \epsilon_{ijk}\ \partial_i a_j\ \dot{a}_k. \tag{21}$$

This Γ^3_+ is exactly the well-known Pontrjagin term (precisely action) under the *temporal* gauge condition: $a_0 = 0$, and the asymptotic behavior (5). We notice that the usual Pontrjagin term is a number-like term with the factor $1/8\pi^2$. However, the above Γ^3_+ is a phase-like term, thus has a different factor π to the usual form. It should be pointed out that the above calculations can be extended to the non-Abelian gauge field interaction without difficulty.

The above results [Eqs. (16) and (21)] lead to a conclusion that for the low-energy gauge interacting system, the asymptotic behavior of the gauge field, based on the consideration of topological boundary, results in an additional term to the effective action of the system: it is the Chern-Simons term for three-dimensional spacetime, and the Pontrjagin term for the four-dimensional spacetime. It is not an accident that the additional term—Chern-Simons term or Pontrjagin term, is a topological invariant, because it comes from the Berry's argument of geometric phase, which is associated with the nontrivial topological structure—holonomy in the parameter space.

According to the above argument of the role of geometric phase in the path integral formalism of quantum theory, we see that in the functional formalism of gauge field theory, the geometric term Γ^2_+ or Γ^3_+ can only appear as an addition term in the $effective$ action, that is

$$S_{eff} = S + \Gamma^i_+, \quad i = 2, 3, \tag{22}$$

where S is the usual "classical" action as in Eq. (6). One may wonder that since Γ^i_+ is a phase-type term, how it can enter the system as a part of S_{eff} that determines the dynamical behavior of the system. This could be answered from two considerations: First, the term Γ^i_+ essentially results from the Eq. (5), a topological requirement, even though it is obtained by employing Berry's scheme. It is actually determined by the gauge field in functional form. Therefore, it can not be simply regarded as a phase. Instead, it takes into account the adiabatically evolution of the system from the point of view of low-energy perturbation. Second, we know that the action S in Eq. (22) is the same a classical functional as Γ^i_+. Even if we treat S_{eff} as a quantum object to quantize it, as many people have done, we see that Γ^i_+ does not contribute to the Lagrangian equations of the quantum fields.

We present several remarks on the above results:

(i) As is well known, the topological term was first introduced in quantum gauge theory by resolving the chiral anomaly in the one-loop triangle diagrams that breaks the usual Ward-Takahashi identities[1]. Here we see that the topological term, based on the low-energy approximation, enters the action automatically. In field theory, it is not difficult to extend the geometric phase from the classical quantity to the quantum quantity. Then the topological term, as a quantum quantity, will be helpful in resolving the anomaly of the chiral current in gauge theory. This will be discussed in the forthcoming paper.

(ii) The presence of Witten's work on topological field theory[3] has recently raised much more interest in theoretical physics. As that interpreted, there still is not a physical realization of the topological term as the dynamical term in action. Therefore, the functional integral of the topological field is called partition function. In our formulation, the topological term is interpreted as a result of the boundary condition of the gauge system, in which the adiabatically condition is required. We know that in the usual gauge field theory, the dynamics of the gauge system is determined by S in Eq. (22). Now we may consider such an extreme situation that the system is confined in the ground state determined by S, which is

highly degenerated and leads to vanish of the average values of the dynamical terms. Then the topological term becomes to dominate the effective action. In this consideration, the topological field theory could be regarded as the ground state reduction of the usual gauge field theory.

(iii) It was revealed recently that the Chern-Simons term plays an important role in the low-dimensional physical world. It brings two-dimensional nontrivial topology[13] to the physical system, and gives rise to some interesting observations, for examples, exotic statistics of quasi-particles[14], the soliton solution[15], and mass-generating of the gauge field[16]. Our formulation of action Eq. (22) implies that the above observations principally exist in the gauge systems in low dimensions, and the induced mass of the gauge field is determined by the coupling constant. The details are omitted here.

In sum, we re-analyze the path integral formalism of the gauge system in this Letter. For the sufficiently low-energy gauge interaction, the asymptotic behavior of the gauge field, based on the consideration of topological boundary, implies a closed loop traced by the gauge field in the time interval: $(-\infty, \ +\infty)$. Adopting Berry's argument of geometric phase, we show that the adiabatic evolution of the gauge system around the loop results in an additional term to the effective action: the Chern-Simons term for three-dimensional spacetime, and the Pontrjagin term for the four-dimensional spacetime. This approach gives an alternative interpretation of the topological terms in physics.

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